Average-case complexity of Maximum Weighted Independent Sets

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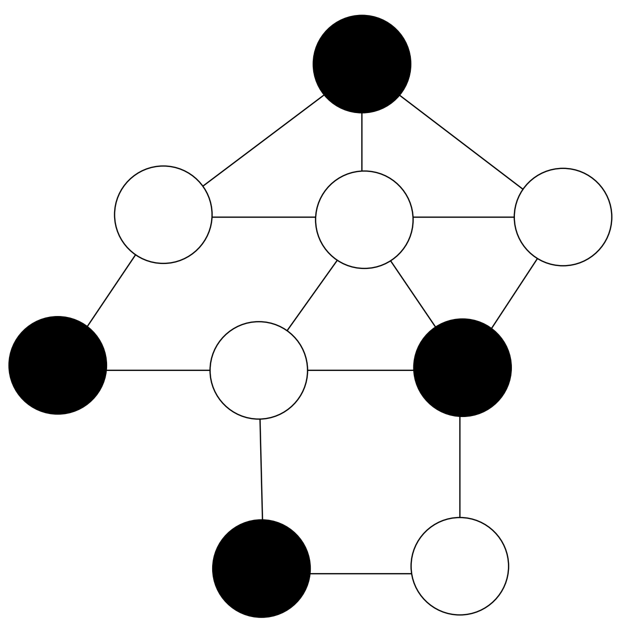
Outline

- Average-case analysis of computational complexity. Independent Sets
- A ‘corrected’ BP algorithm: the cavity expansion
- Results: sufficient condition, hardness results.
- Conclusion

Combinatorial Optimization with Random Costs

- Goal: Study relation between randomness and computational complexity
- Problems of interest: combinatorial optimization on graph - here: Maximum Weighted Independent Set
- Rather than random graph, random costs
- Identify relations between graph structure, cost distribution, and complexity
- Techniques used: 'message-passing' algorithm, correlation decay analysis.

Max Weight Independent Sets



- Graph (V,E), weights $\mathbf{W} \in \mathbb{R}_+^{|V|}$
- Independent Set U: $\forall u, v \in U, (u,v) \notin E$
- Max-Weight Independent Set (MWIS): given weights W, find U which maximizes $\sum_{v \in U} W_v$
- Our setting: weights are random i.i.d variables from a joint distribution F
- Arbitrary graph of bounded degree Δ
- Similar models in Gamarnik, Nowicki, Swircz [05], Sanghavi, Shah, Willsky [08]

Hardness facts

- NP-hard, even for $\Delta = 3$
- Poly-time approx algorithm of ratio α : finds an IS $\tilde{U}$ such that $\frac{W(U)}{W(\tilde{U})} < \alpha$
- Poly-time Approximation Scheme: for all $\alpha > 1$, there exists a approx. algorithm of ratio α
- Hastad [99] NP-hard to approximate within $$n^\beta, \beta < 1$$
- Trevisan [01] NP-hard to approximate within $$\frac{\Delta}{2^{O(\sqrt{\log \Delta})}}$$

A first result

Theorem: Assume $\mathbb{P}(W > t) = \exp(-t)$, $\Delta \leq 3$

The problem can be approximated in polynomial time: for any $\epsilon > 0$, in $O(|V| 2^{\epsilon^{-2}})$, there exists an algo. which finds an I.S. I such that

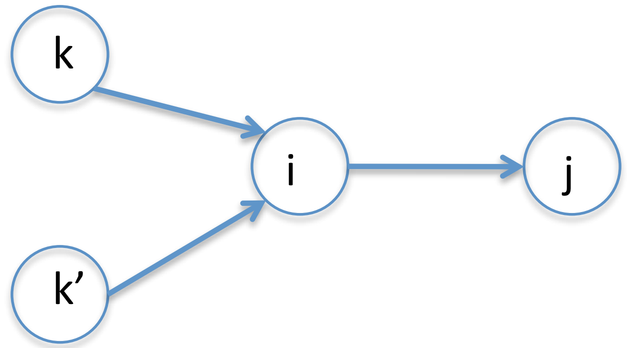
$$\mathbb{P}(\frac{W(I^*)}{W(I)} > 1 + \epsilon) < \epsilon$$

* Linear in |V| (with parallel computation, constant computation time)

* Case $\Delta \leq 3$ exceptional?

* Case of Exponential weights exceptional?
~ Only distribution which works?
~ MWIS always easy with random weights?

Message passing for MWIS



- Graphical model formulation of MWIS:

$$p(x) = \frac{1}{Z} \prod_{i,j \in E} \mathbf{1}_{\{x_i + x_j \le 1\}} \prod_{i \in V} \exp(w_i x_i)$$

- Max-product (BP):

$$\mu_{i \to j}(0) = \max \left\{ \prod_{k \in \mathcal{N}_i, k \neq i} \mu_{k \to i}(0), e^{w_i} \prod_{k \in \mathcal{N}_i, k \neq i} \mu_{k \to i}(1) \right\}$$

$$\mu_{i \to j}(1) = \prod_{k \in \mathcal{N}_i, k \neq i} \mu_{k \to i}(0)$$

set $M_{i \to j} = \log\left(\frac{\mu_{i \to j}(0)}{\mu_{i \to j}(1)}\right)$

then: $M_{i \to j} = \max(0, W_i - \sum_{k \in \mathcal{N}_i, k \neq j} M_{k \to j})$

LP relaxation for MWIS - connection with BP

- IP formulation of MWIS:

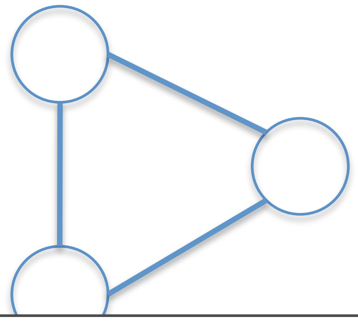
$$\max_{\mathbf{x}} \sum_i W_i x_i$$
$$\text{s.t. } \forall (i,j) \in E, x_i + x_j \leq 1$$
$$\forall i, x_i \in \{0,1\}$$

- LP relaxation:

$$\max_{\mathbf{x}} \sum_i W_i x_i$$
$$\text{s.t. } \forall (i,j) \in E, x_i + x_j \leq 1$$
$$\forall i, 0 \leq x_i \leq 1$$

- LP is tight at variable i if $x_i \in \{0,1\}$
- Fact [Sanghavi, Shah, Willsky]: If BP converges at variable i, then the LP is tight at i
- Converse: if the LP is not tight, then BP does not converge

IP solution: one node, opt. cost: 1

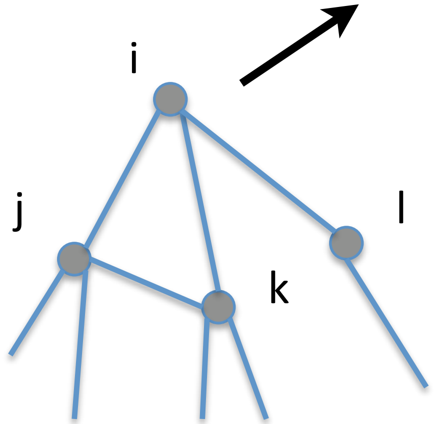


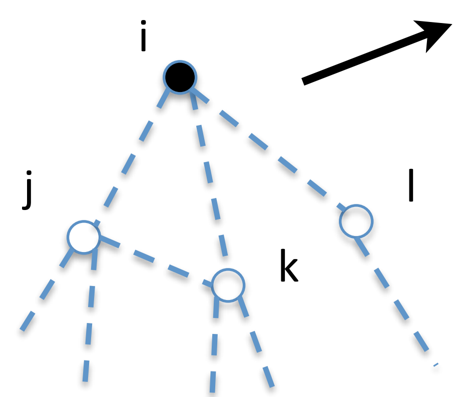
LP solution: (1/2,1/2,1/2), opt. cost: 3/2>1 : LP not tight

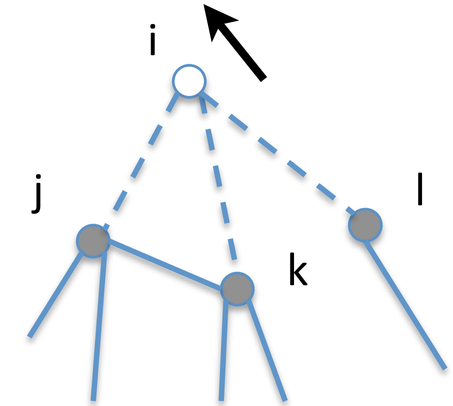
The Cavity Expansion: a corrected BP

–We try to compute exactly $B_G(i) = W(I^*_G) - W(I^*_{G\setminus\{i\}})$
if >0, then $i \in I^*_G$, otherwise $i \notin I^*_G$ (w.p.1)

$$W(I^*_G) = \max(W_i + W(I^*_{G\setminus\{i,j,k,l\}}, W(I^*_{G\setminus\{i\}})$$



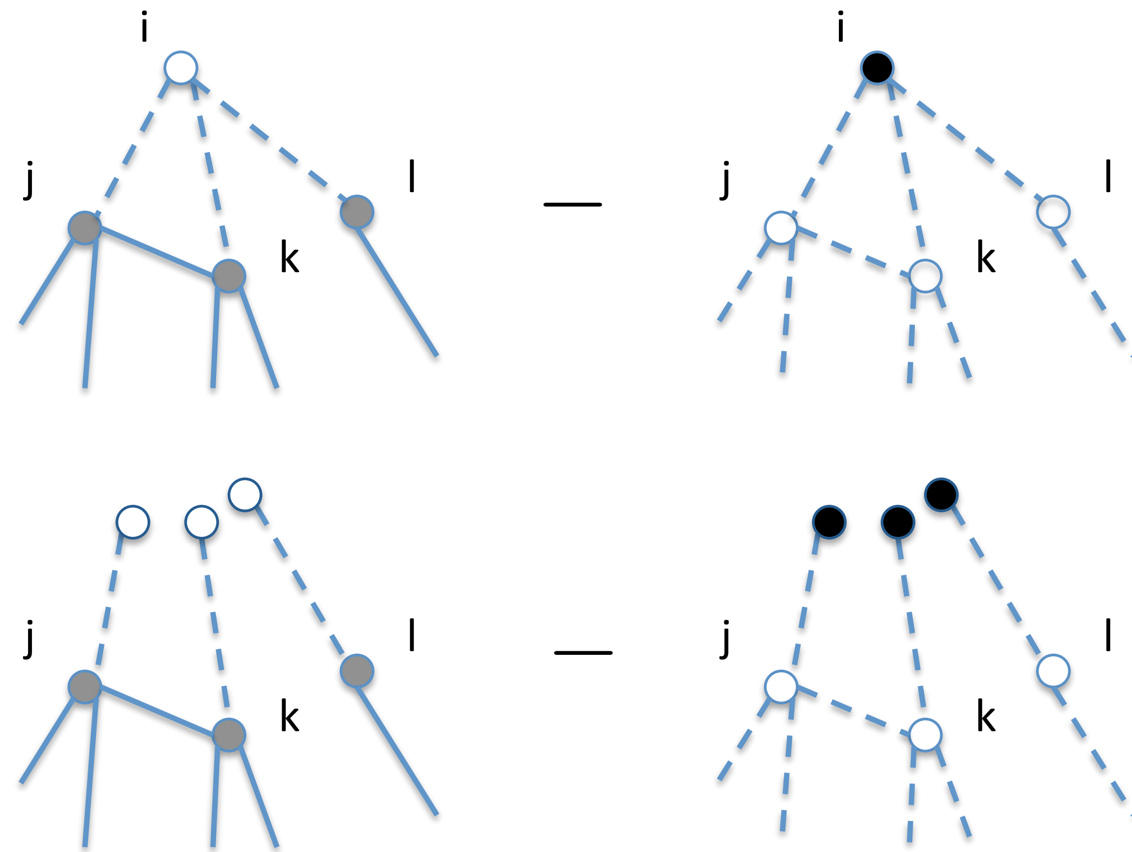




$$- \quad W(I^*_{G\setminus\{i\}})$$

The Cavity Expansion: a corrected BP

– So: $B_G(i) = \max\Big(0, W_i - \big(W(I^*_{G\setminus\{i\}}) - (W(I^*_{G\setminus\{i,j,k,l\}})\big)\Big)$



The Cavity Expansion: a corrected BP

$$W(I^*_{G\setminus\{i\}}) - W(I^*_{G\setminus\{i,j,k,l\}}) =$$

$$W(I^*_{G\setminus\{i\}}) - W(I^*_{G\setminus\{i,j\}}) \quad +$$

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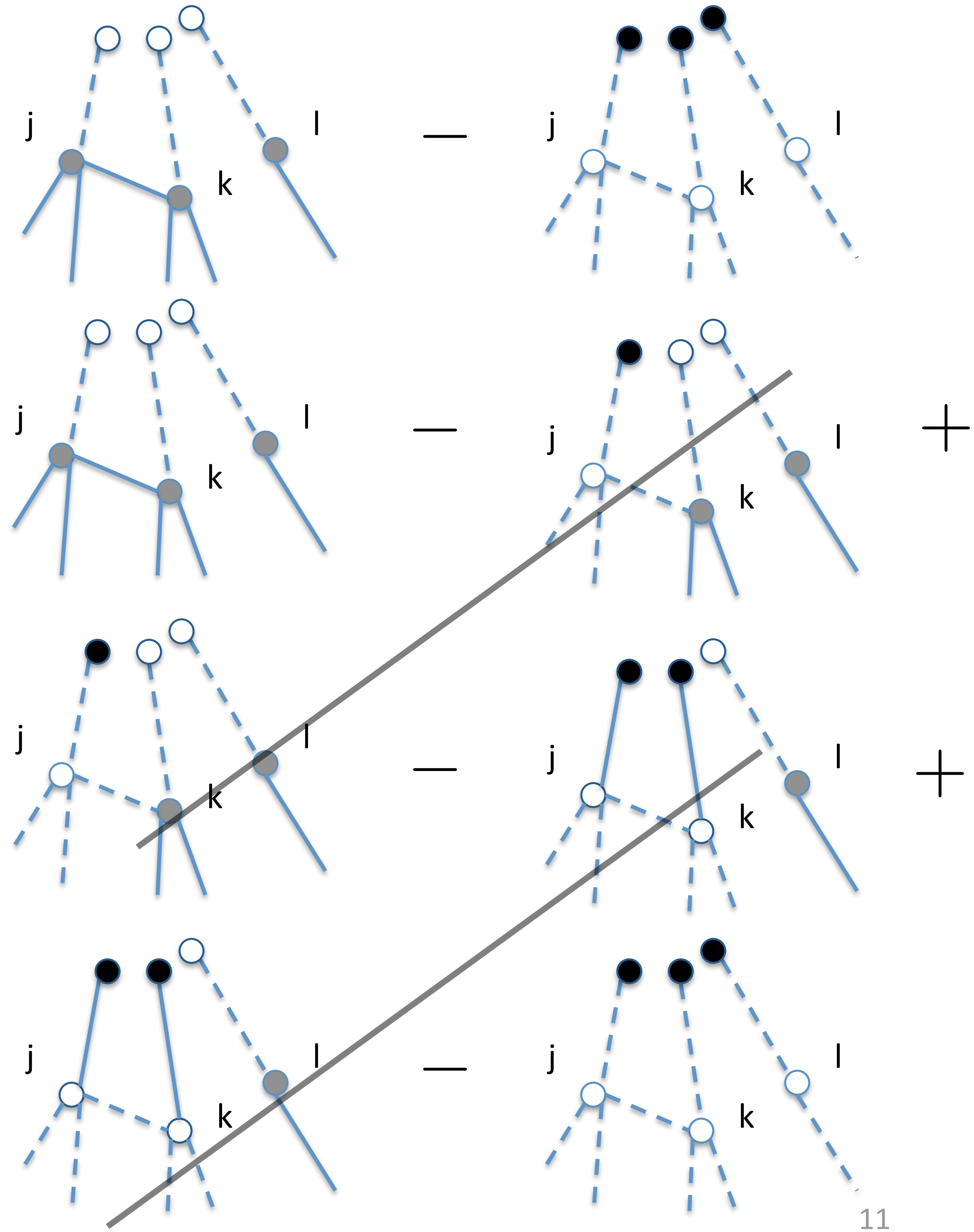

The Cavity Expansion: a corrected BP

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$$\left(= B_{G\setminus\{i\}}(j) \right)$$

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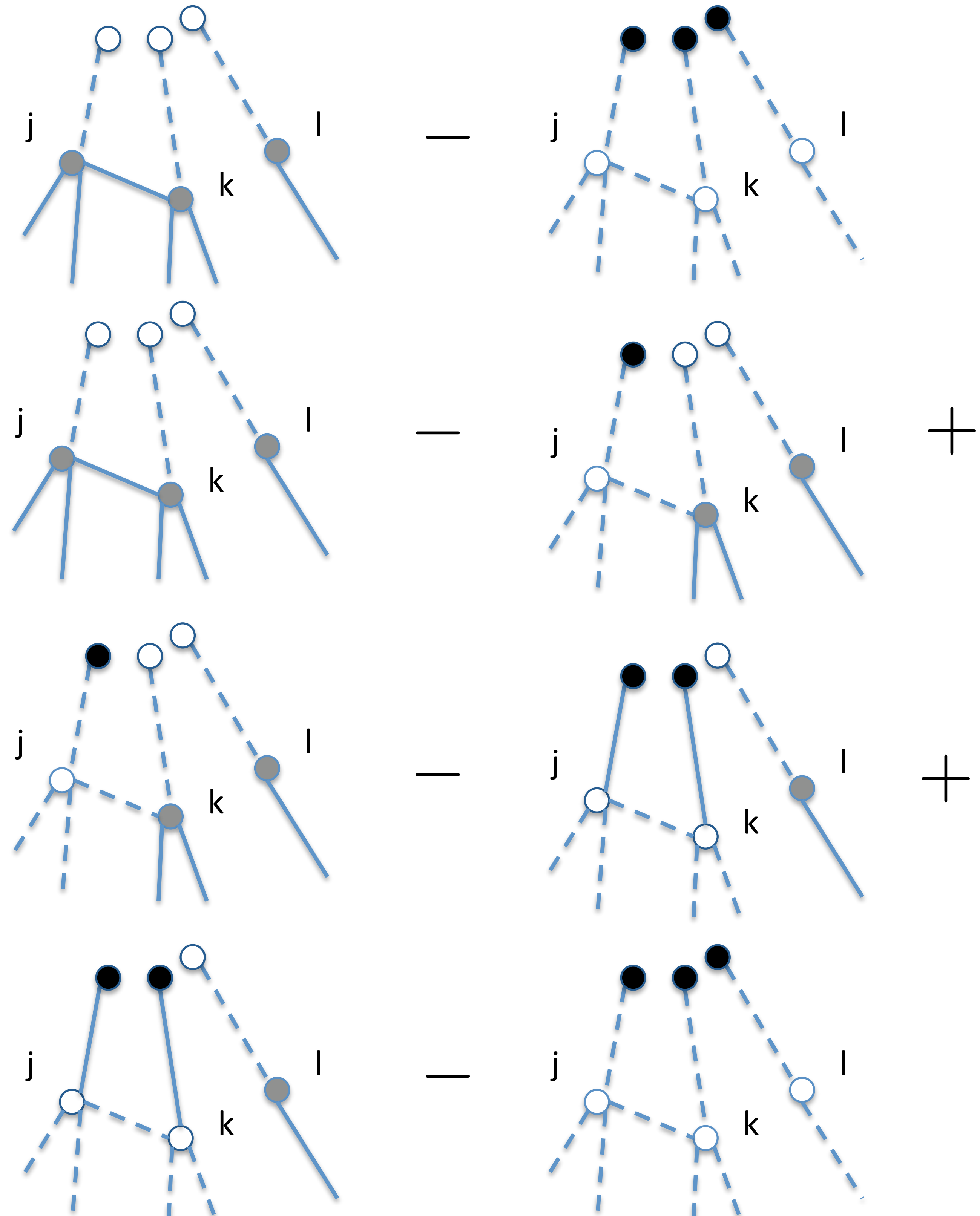


Cavity Expansion: Summary

- Cavity Expansion (for IS):

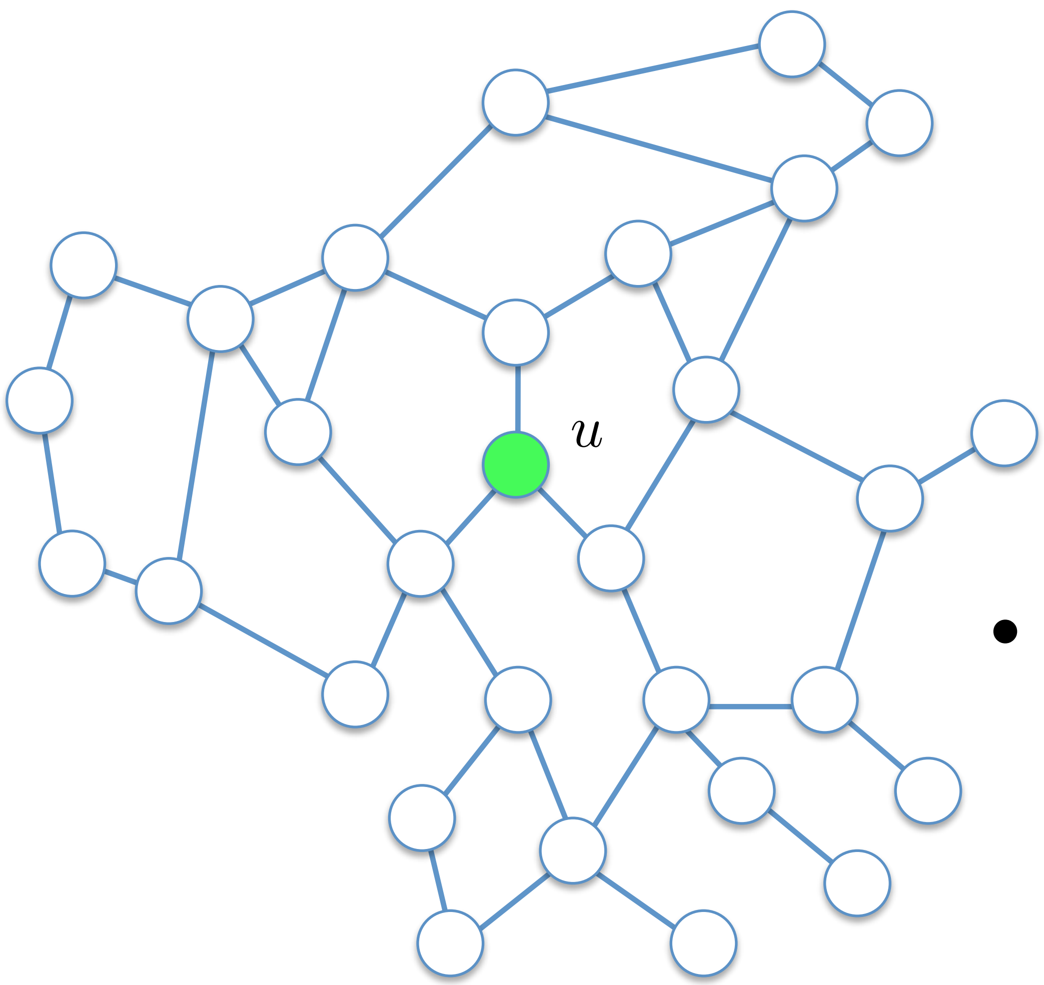
$$B_G(i) = \max(0, W_i - B_{G\setminus\{i\}}(j) - B_{G\setminus\{i,j\}}(k) - B_{G\setminus\{i,j,k\}}(l))$$

- BP (for IS):

$$M_G(i) = \max(0, W_i - M_G(j) - M_G(k) - M_G(l))$$

- Generalization for arbitrary optimization
- Similar approaches (for counting): Weitz (06), Bayati,Gamarnik,Katz, Nair, Tetali (07), Jung and Shah (07)
- CE always converges, and is correct at termination
- caveat: running time $O(\Delta^{|V|})$
- Fix: interrupt after a fixed number of iterations t

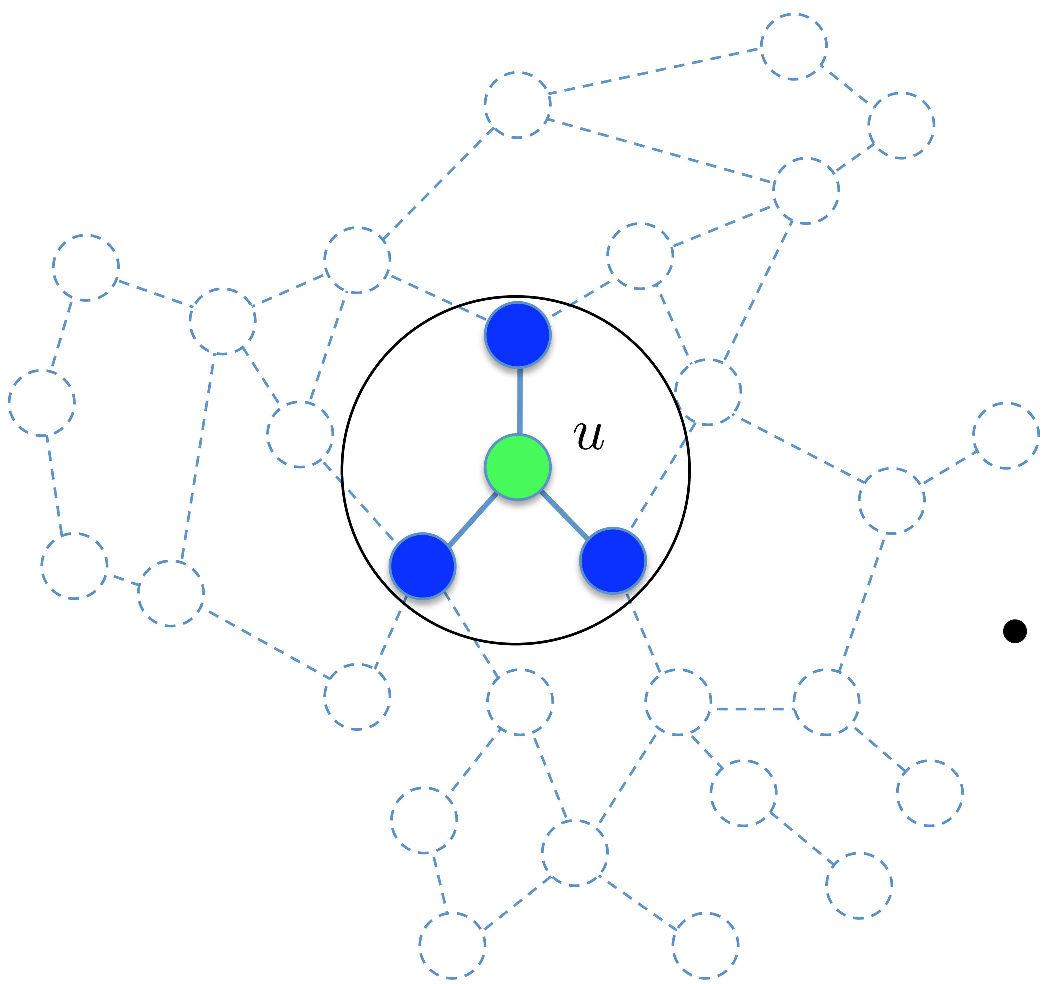
Correlation Decay analysis



- Let $B_G^r(i)$ be the r-step approx of $B_G(i)$
- Definition: System exhibits correlation decay if
$$|B_G^r(i) - B_G(i)| \to 0$$
exponentially fast (in r)

- Implies: wether u is in the MWIS is asymptotically independent of the graph beyond a certain boundary

Correlation Decay analysis



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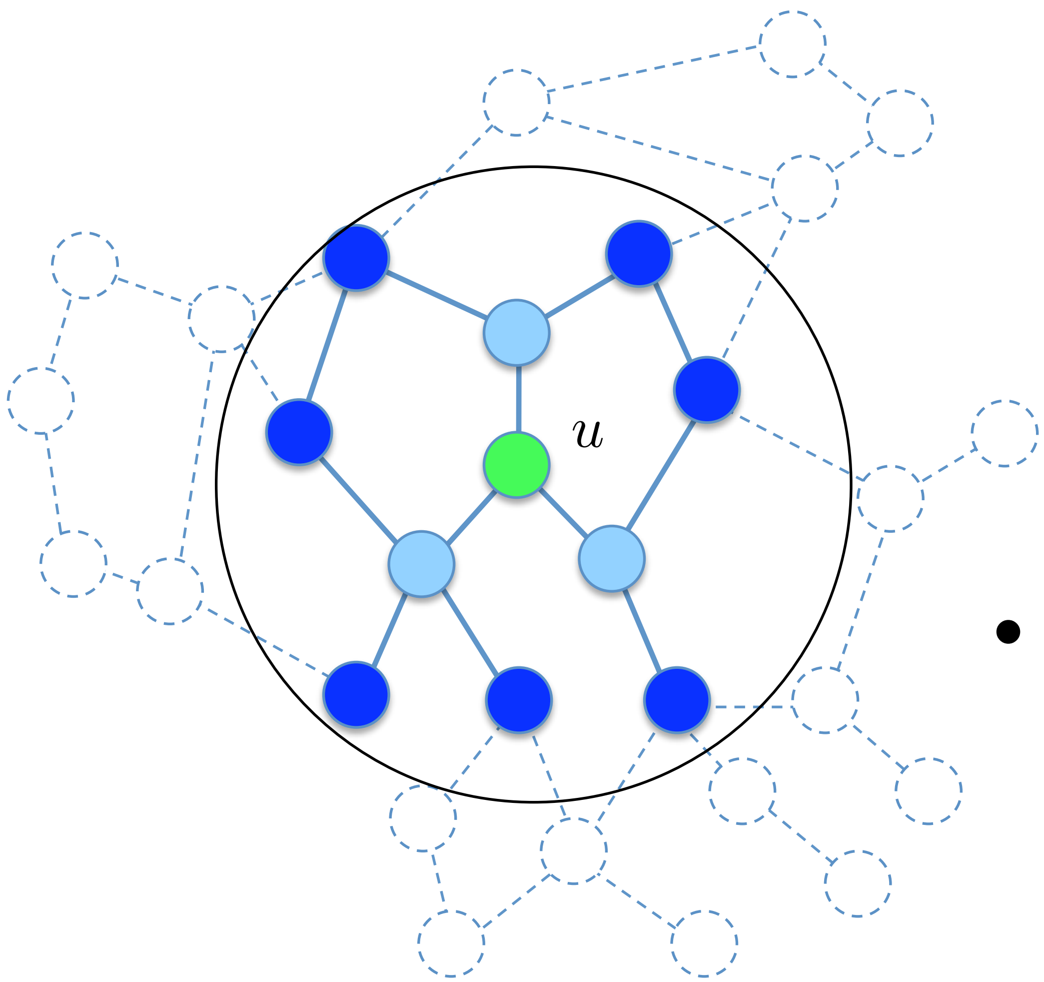

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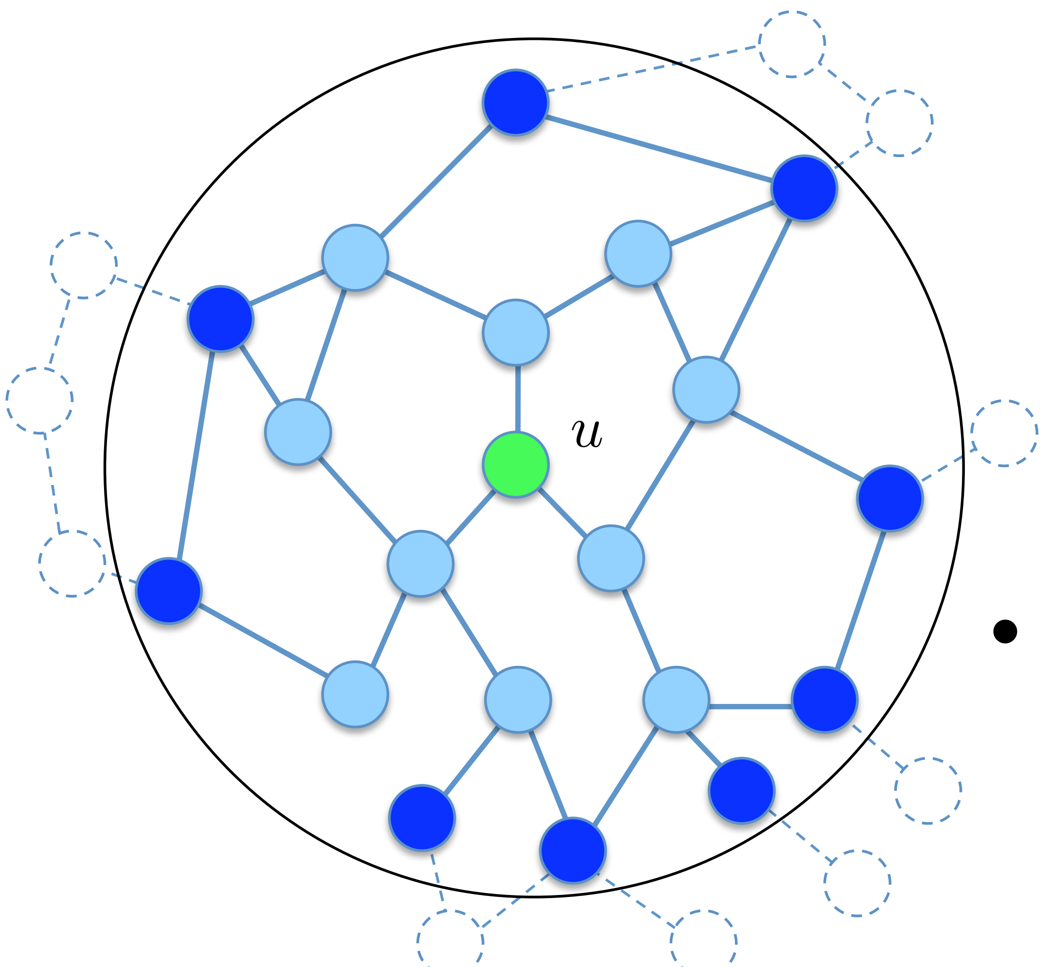


- Let $B^r_G(i)$ be the r-step approx of $B_G(i)$
- Definition: System exhibits correlation decay if
$$|B^r_G(i) - B_G(i)| \to 0$$
exponentially fast (in r)

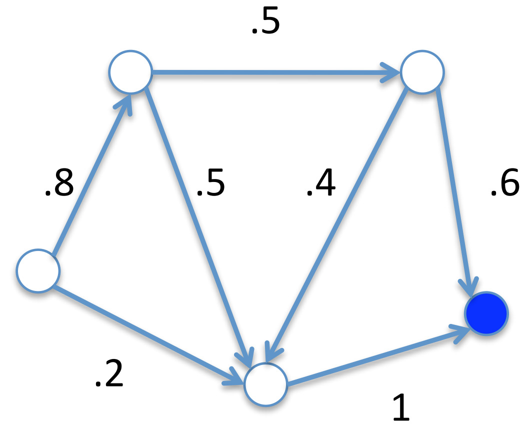
- Implies: wether u is in the MWIS is asymptotically independent of the graph beyond a certain boundary
- Recall $I^* = \{i : B_G(i) > 0\}$
- Candidate solution:
$$I^r = \{i : B^r_G(i) > 0\}$$

Proof sketch of near-optimality

- Introduce 'Lyapunov' function $L_G(i) = \mathbb{E}[\exp(-B_G(i))]$
- From CE and expo weights assumption, find a recursion on the $L_G(i)$: $L_G(i) = 1 - 1/2(L_{G\setminus\{i\}(j)} L_{G\setminus\{i,j\}}(k))$
- This implies a non-expansion of the recursion of L_G
- Prune a small fraction δ of the nodes
- This implies a contraction of factor $(1-\delta)$
- After r steps, error is $(1-\delta)^r + \delta$
- minimize delta as a function of r => correlation decay
- Final steps: prove that if $B_G^r(i) \approx B_G(i)$, then $I^r \approx I^*$

Generalization

- Phase-type distribution: absorption time in a Markov Process with exponential transit times



- Dense in the space of all distributions
- Different Lyapunov function to analyze recursions
- For any phase-type distribution F, can compute $\alpha(F)$ such that if $\alpha(F)\Delta < 1$, corr. decay occurs.
- Not many distributions work with $\Delta \geq 2$

Theorem: assume $\mathbb{P}(W > t) = \frac{1}{\bar{\Delta}} \sum_i \exp(-\rho^i t) \qquad \rho > 17 \qquad \Delta \leq \bar{\Delta}$
Then corr. decay occurs, average optimization easy

Negative result

$$\mathbb{P}(W > t) = \exp(-t) \qquad \Delta \leq \Delta^*$$

Unless P=NP, the problem cannot be solved in polynomial time

Proof Intuition:

How good of a MIS is the random MWIS?

$$\frac{I^*_{\text{MIS}}}{E[I^*_{\text{MWIS}}]} \leq O(\log \Delta)$$

But MIS is inapproximable within $\frac{\Delta}{2^{O(\sqrt{\log \Delta})}}$

Conclusion

- New algorithm for optimization in sparse graphs
- Long range-independence implies existence of efficient and distributed algo
- Open Q:
  - Relation between long-range dependence and hardness?
  - Pseudo-random cost and long-range independence?
  - Polytope interpretation (average integrality gap?)